

Spectral action for a one-parameter family of Dirac operators on SU(2) and SU(3)

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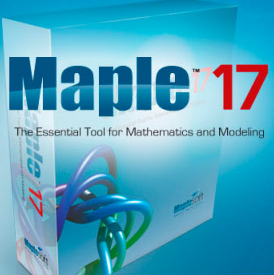
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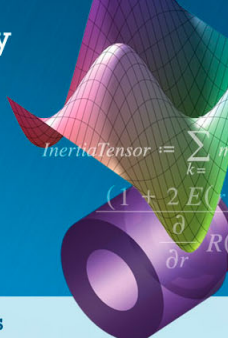
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Spectral action for a one-parameter family of Dirac operators on $SU(2)$ and $SU(3)$

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The one-parameter family of Dirac operators containing the Levi-Civita, cubic, and the trivial Dirac operators on a compact Lie group is analyzed. The spectra for the one-parameter family of Dirac Laplacians on $SU(2)$ and $SU(3)$ are computed by considering a diagonally embedded Casimir operator. Then the asymptotic expansions of the spectral actions for $SU(2)$ and $SU(3)$ are computed, using the Poisson summation formula and the two-dimensional Euler-Maclaurin formula, respectively. The inflation potential and slow-roll parameters for the corresponding pure gravity inflationary theory are generated, using the full asymptotic expansion of the spectral action on $SU(2)$. © 2013 American Institute of Physics. [<http://dx.doi.org/10.1063/1.4790484>]

I. INTRODUCTION

On a compact Lie group, one may naturally associate a one-parameter family of Dirac operators by interpolating the torsion of a connection given by the Lie bracket. This family of Dirac operators, is parametrized by a real parameter, t . For $t = 1/2$ one obtains the geometric spin Dirac operator, algebraic cubic Dirac operator of Kostant when $t = 1/3$,⁸ and the trivial Dirac operator when $t = 0$. The trivial Dirac operator is used in a model of loop quantum gravity (LQG) to model tetrads.¹ While such a generalized family of operators is mathematically interesting, the authors' original motivation for considering this family was to attempt to calculate the spectral action of the LQG spectral triple of Aastrup, Grimstrup, and Nest.^{1,10} The results obtained in this paper are the asymptotic expansions of the spectral actions for $SU(2)$, $SU(3)$, and $U(1) \times SU(2)$. Given an energy scale Λ , the spectral action counts the number of eigenstates with energy below Λ . As Λ grows to a large scale, the coefficients of the Λ -asymptotic of the spectral action capture the residues of the noncommutative Zeta function.⁴ The spectral action serves as an action functional in noncommutative geometry. These results may serve as a step toward a LQG spectral action computation (for $t = 0$), also coupled to matter.

The paper is organized in the following manner. Section I defines the one-parameter family of Dirac operators that we are considering. In Sec. II, we express the Dirac Laplacian action on any Clifford module in terms of the action of the Lie algebra's Casimir element on finite-dimensional irreducible representations of the Lie group. This interpretation reduces the problem of finding the spectrum on a general compact Lie group to the specific Clebsch-Gordan decomposition of the tensor products of an irreducible representation with the Weyl representation. We calculate the spectrum of the one-parameter family of Dirac Laplacians on $SU(2)$ and the asymptotic expansion of their spectral actions on in Sec. III. In Sec. IV, we apply the developed machinery to compute the Dirac Laplacian spectrum for $SU(3)$, and in Sec. V, we compute the asymptotic expansion of the spectral action for $SU(3)$ using the two-dimensional Euler-Maclaurin formula, to constant order in Λ . In Sec. VI, we consider a toy inflation model of the Friedman universe using results developed in Sec. III. We end the paper with an appendix that details the calculations involved in the two-dimensional Euler-Maclaurin formula for the $SU(3)$ spectral action calculation.

II. ONE-PARAMETER FAMILY OF DIRAC OPERATORS \mathcal{D}_t

In this section, we follow Agricola² to define a family of Dirac operators on a compact Lie group G , which interpolates the Levi-Civita, cubic, and trivial Dirac operators.

Consider the Lie algebra \mathfrak{g} of G of tangent vector fields on G , invariant under left-multiplication. Then one obtains a family of connections

$$\nabla_X^t := \nabla_X^0 + t[X, \cdot],$$

where ∇^0 is the trivial connection with respect to the left trivialization, and ∇^1 is the trivial connection with respect to the right trivialization.

Let $\langle \cdot, \cdot \rangle$ denote a positive-definite invariant metric on \mathfrak{g} . One checks that ∇^t is a metric $\mathfrak{so}(\mathfrak{g})$ connection, and the torsion, $T(X, Y) = (2t - 1)[X, Y]$, vanishes when $t = 1/2$. Hence the Levi-Civita connection, $\nabla^{1/2}$, is the connection in the linear interpolation halfway between the left trivial connection and the right trivial connection.

The $\mathfrak{so}(\mathfrak{g})$ connection ∇^t lifts to a metric $\widehat{\mathfrak{spin}}(\mathfrak{g})$ connection $\widehat{\nabla}^t$ ¹⁴ given by the formula

$$\widehat{\nabla}_X^t = \nabla_X^0 + t \frac{1}{4} \sum_{k,l} \langle X, [X_k, X_l] \rangle X_k X_l.$$

Let $\mathbb{C} \mathfrak{l}(\mathfrak{g})$ denote the Clifford algebra generated by \mathfrak{g} with the relation $XY + YX = -2\langle X, Y \rangle$ for $X, Y \in \mathfrak{g}$. Let $\{X_i\}$ denote the set of orthonormal basis of \mathfrak{g} with respect to the metric $\langle \cdot, \cdot \rangle$. Let us define the Dirac operator in $\mathbb{C} \mathfrak{l}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ given by $\widehat{\nabla}^t$ to be

$$\mathcal{D}_t := \sum_i X_i \otimes X_i + tH \in \mathbb{C} \mathfrak{l}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}), \quad (1)$$

where

$$H := \frac{1}{4} \sum_{j,k,l} X_j X_k X_l \otimes \langle X_j, [X_k, X_l] \rangle \in \mathbb{C} \mathfrak{l}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}).$$

Since the connection is metric, and it is torsion free at $t = 1/2$, $\mathcal{D}_{1/2}$ is the Levi-Civita Dirac operator. The operator \mathcal{D}_0 is the trivial Dirac operator defined by the trivial connection with respect to the left trivialization. It is the Dirac operator used as a basic construction block of the spectral triple of Aastrup, Grimstrup, and Nest in loop quantum gravity to model tetrads.^{1,10} The Dirac operator, $\mathcal{D}_{1/3}$, with parameter $t = 1/3$ is the cubic Dirac operator of Kostant,⁹ whose square, we will see in a moment, has the simple property of consisting only of its degree two term and its degree zero term. Therefore, the family of Dirac operators \mathcal{D}_t interpolates the most important Dirac operators one considers on a Lie group.

III. SPECTRUM OF \mathcal{D}_t^2

We analyze the Dirac Laplacian \mathcal{D}_t^2 and find that it can be written in terms of the Casimir operator Cas . As the action of Cas on (finite dimensional) irreducible components of a Lie algebra representation is known, we reduce the study of \mathcal{D}_t^2 to the study of Cas action on irreducible components, and give an expression of the spectrum of the Dirac Laplacian \mathcal{D}_t^2 in terms of Lie algebra representations.

The calculation for \mathcal{D}_t^2 is done in Ref. 2. To be self-contained, we include the calculation for our simplified case here.

We calculate

$$\begin{aligned} \mathcal{D}_t^2 &= \left(\sum_i X_i \otimes X_i + tH \right)^2 \\ &= \left(\sum_i X_i \otimes X_i \right)^2 + t \left(\sum_i X_i \otimes X_i \right) H + tH \left(\sum_i X_i \otimes X_i \right) + (tH)^2. \end{aligned}$$

The first term is

$$\begin{aligned} \left(\sum_i X_i \otimes X_i \right)^2 &= \sum_i X_i X_i \otimes X_i X_i + \sum_{i \neq j} X_i X_j \otimes X_i X_j \\ &= \sum_i -1 \otimes X_i^2 + \frac{1}{2} \sum_{i \neq j} X_i X_j \otimes X_i X_j - \frac{1}{2} \sum_{i \neq j} X_i X_j \otimes X_j X_i \\ &= 1 \otimes \text{Cas} + \frac{1}{2} \sum_{ij} X_i X_j \otimes [X_i, X_j]. \end{aligned}$$

For the cross term, first observe that for $i \neq j, k, l$,

$$X_i X_j X_k X_l + X_j X_k X_l X_i = 0$$

by the Clifford relation. Therefore, i must equal one of j, k, l .

For $i = j$,

$$\begin{aligned} \sum_{i,j,k,l} (X_i X_j X_k X_l + X_j X_k X_l X_i) \otimes \langle [X_j, X_k], X_l \rangle X_j &= \sum_{j,k,l} -2X_k X_l \otimes \langle [X_j, X_k], X_l \rangle X_j \\ &= \sum_{k,l} -2X_k X_l \otimes [X_k, X_l]. \end{aligned}$$

The same expression is obtained for $i = k$ and $i = l$, thus summing over the three possibilities, one obtains

$$\begin{aligned} t \left(\sum_i X_i \otimes X_i \right) H + t H \left(\sum_i X_i \otimes X_i \right) &= t \frac{1}{4} \sum_{i,j,k,l} (X_i X_j X_k X_l + X_j X_k X_l X_i) \otimes \langle [X_j, X_k], X_l \rangle X_i \\ &= -3t \frac{1}{2} \sum_{k,l} X_k X_l \otimes [X_k, X_l]. \end{aligned}$$

For the scalar term

$$(tH)^2 = t^2 \frac{1}{16} \sum_{b,c,d,j,k,l} X_b X_c X_d X_j X_k X_l \otimes \langle [X_b, X_c], X_d \rangle \langle [X_j, X_k], X_l \rangle. \quad (2)$$

If $b \neq j, k$, or l , then the above expression (2) is zero by the Clifford relation.

For $b = j, k$, or l ,

$$\sum_{b,c,d,j,k,l} X_b X_c X_d X_j X_k X_l \otimes \langle [X_b, X_c], X_d \rangle \langle [X_j, X_k], X_l \rangle, \quad (3)$$

$$= -3 \sum_{c,d,k,l} X_c X_d X_k X_l \otimes \langle [X_c, X_d], [X_k, X_l] \rangle. \quad (4)$$

If $c \neq k$ or l , then the above expression (3) is zero by the Clifford relation and Jacobi identity. When $c = k$, or l , b has to equal l , or k . Thus

$$\begin{aligned} -3 \sum_{c,d,k,l} X_c X_d X_k X_l \otimes \langle [X_c, X_d], [X_k, X_l] \rangle &= 6 \sum_{k,l} \langle [X_k, X_l], [X_k, X_l] \rangle \\ &= 6 \sum_{k,l} \langle X_k, -[X_l, [X_l, X_k]] \rangle \\ &= 6 \sum_k \langle X_k, \text{ad}_{\text{Cas}} X_k \rangle \\ &= 6 \text{Tr}(\text{ad}_{\text{Cas}}). \end{aligned}$$

Together with Kostant's generalized formula of Freudenthal and de Vries,⁸

$$\frac{1}{24} \text{Tr}(\text{ad}_{\text{Cas}}) = |\rho|^2, \quad (5)$$

the scalar term (2) is found to be

$$\begin{aligned} (tH)^2 &= t^2 \frac{3}{8} \text{Tr}(\text{ad}_{\text{Cas}}) \\ &= 9t^2 |\rho|^2. \end{aligned}$$

Therefore, the total expression of \mathcal{D}_t^2 can be written as the following.

Proposition 3.1 (Ref. 2): Let $\{X_i\}$ be the set of orthonormal basis of \mathfrak{g} , \mathcal{D}_t to be defined as in Eq. (1). Then

$$\mathcal{D}_t^2 = 1 \otimes \text{Cas} + (1 - 3t) \frac{1}{2} \sum_{k,l} X_k X_l \otimes [X_k, X_l] + 9t^2 |\rho|^2.$$

Notice that for $t = 1/3$, the degree one term vanishes. $\mathcal{D}_{1/3}$ is called the cubic Dirac operator of Kostant.⁹

The degree two term is written as a Casimir, however the degree one term is still rather complicated. As it turns out, we can further simplify the expression and write the degree one term in terms of Casimir.

Theorem 3.2 (Ref. 8): The map $\pi : \mathfrak{g} \rightarrow \mathbb{C} \text{l}(\mathfrak{g})$ given in the orthonormal basis $X_k \in \mathfrak{g}$ by

$$\pi(X_i) := \frac{1}{4} \sum_{k,l} \langle X_i, [X_k, X_l] \rangle X_k X_l$$

is a Lie algebra homomorphism.

Since the Lie algebra is semi-simple, π is in fact an injection.

We extend π to $\pi : \mathcal{U}(\mathfrak{g}) \rightarrow \mathbb{C} \text{l}(\mathfrak{g})$. Now the degree one term in \mathcal{D}_t^2 can be written as

$$\begin{aligned} \frac{1}{2} \sum_{k,l} X_k X_l \otimes [X_k, X_l] &= \sum_i \frac{1}{2} \sum_{k,l} \langle [X_k, X_l], X_i \rangle X_k X_l \otimes X_i \\ &= 2 \sum_i \pi(X_i) \otimes X_i. \end{aligned}$$

Let $\Delta : \mathcal{U}(\mathfrak{g}) \rightarrow \mathcal{U}(\mathfrak{g} \oplus \mathfrak{g}) = \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ denote the co-multiplication given by the diagonal embedding

$$\Delta(X_i) := X_i \otimes 1 + 1 \otimes X_i \in \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}).$$

Then

$$\begin{aligned} \Delta \text{Cas} &= - \sum_i (X_i \otimes 1 + 1 \otimes X_i)^2 \\ &= 1 \otimes \text{Cas} + \text{Cas} \otimes 1 - 2 \sum_i X_i \otimes X_i. \end{aligned}$$

Thus,

$$-\frac{1}{2} \sum_{k,l} X_k X_l \otimes [X_k, X_l] = (\pi \otimes 1)(\Delta \text{Cas} - 1 \otimes \text{Cas} - \text{Cas} \otimes 1).$$

We obtain

Theorem 3.3: For \mathcal{D}_t defined as in Eq. (1), \mathcal{D}_t^2 can be written as

$$\mathcal{D}_t^2 = (\pi \otimes 1)T_t$$

where

$$\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}) \ni T_t := 1 \otimes \text{Cas} + (3t - 1)(\Delta \text{Cas} - 1 \otimes \text{Cas} - \text{Cas} \otimes 1) + 9t^2|\rho|^2.$$

Hence, by the injectivity of π , the action of \mathcal{D}_t^2 can be deduced from the action of T_t , and T_t is written as a combination of $1 \otimes \text{Cas}$, $\text{Cas} \otimes 1$, and ΔCas .

Lemma 3.4 (Ref. 7): *Let V_λ be an irreducible representation of \mathfrak{g} with highest weight λ . Then Cas acts as a scalar on V_λ , and the scalar is given by*

$$|\lambda + \rho|^2 - |\rho|^2 = \langle \lambda + 2\rho, \lambda \rangle.$$

Suppose that \mathcal{S} is a Clifford $\mathbb{C} \mathfrak{l}(\mathfrak{g})$ -module so that $\mathbb{C} \mathfrak{l}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ acts on $\mathcal{S} \otimes L^2(G)$, where $\mathbb{C} \mathfrak{l}(\mathfrak{g})$ acts on \mathcal{S} via the Clifford action and $\mathcal{U}(\mathfrak{g})$ acts on $L^2(G)$ as left-invariant differentiation.

By Peter-Weyl, $L^2(G)$ decomposes as the norm closure of

$$\bigoplus_{\lambda \in \widehat{G}} V_\lambda \otimes V_\lambda^*,$$

where λ ranges over the irreducible representations \widehat{G} of G .

Since $\mathcal{U}(\mathfrak{g})$ acts as left invariant differential operators on $L^2(G)$, it acts as the identity on the dual components V_λ^* .

Theorem 3.5 (Ref. 8): *Let \mathcal{S} be any $\mathbb{C} \mathfrak{l}(\mathfrak{g})$ -module. Then the \mathfrak{g} -representation on \mathcal{S} defined by composition with π is a direct sum of ρ -representations, where ρ is the half sum of all positive roots, the Weyl vector.*

The theorem implies that as an $\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ representation, $\mathcal{S} \otimes L^2(G)$ decomposes as

$$\mathcal{S} \otimes L^2(G) = \bigoplus_{\rho} \bigoplus_{\lambda \in \widehat{G}} V_\rho \otimes V_\lambda \otimes V_\lambda^*.$$

The sum \bigoplus_{ρ} and the dual components V_λ^* only change the multiplicity of the action. We will for the moment ignore them and only look at the $V_\rho \otimes V_\lambda$ part.

We can read off how $\text{Cas} \otimes 1$ and $1 \otimes \text{Cas}$ act on $V_\rho \otimes V_\lambda$ already by directly applying Lemma 3.4. However to know how T_t , hence \mathcal{D}_t^2 , acts on $V_\rho \otimes V_\lambda$, we need to know the action of ΔCas , which can be obtained if one knows the direct sum decomposition of $V_\rho \otimes V_\lambda$ into irreducible components, the so-called Clebsch-Gordan decomposition. In effect, we are reducing the study of the spectrum of \mathcal{D}_t^2 to the Clebsch-Gordan decomposition of $V_\rho \otimes V_\lambda$.

Suppose that $V_\rho \otimes V_\lambda$ decomposes as

$$V_\rho \otimes V_\lambda = \bigoplus_{\gamma} V_{\lambda \pm \gamma} \quad (6)$$

for some weights γ .

We examine on it the action of

$$T_t = 1 \otimes \text{Cas} + (3t - 1)(\Delta \text{Cas} - 1 \otimes \text{Cas} - \text{Cas} \otimes 1) + 9t^2|\rho|^2.$$

One has

$$\begin{aligned} \left. (- (3t - 1) \text{Cas} \otimes 1 + 9t^2|\rho|^2) \right|_{V_\rho \otimes V_\lambda} &= - (3t - 1) \langle \rho + 2\rho, \rho \rangle + 9t^2|\rho|^2 \\ &= (3t - 1)(3t - 2)|\rho|^2 + |\rho|^2. \end{aligned}$$

and

$$\begin{aligned} \left. ((3t - 1)\Delta \text{Cas} - 1 \otimes \text{Cas}) \right|_{V_\rho \otimes V_\lambda} &= (3t - 1) \left(\langle \lambda \pm \gamma + 2\rho, \lambda \pm \gamma \rangle - \langle \lambda + 2\rho, \lambda \rangle \right) \\ &= 2(3t - 1) \langle \lambda + \rho, \pm \gamma \rangle + (3t - 1)|\gamma|^2 \\ &= 2 \langle \lambda + \rho, \pm (3t - 1)\gamma \rangle + |(3t - 1)\gamma|^2 - (3t - 1)(3t - 2)|\gamma|^2. \end{aligned}$$

Thus, the action of T_t on $V_\rho \otimes V_\lambda$ is given by the following.

Theorem 3.6: *Let γ denote the weights in the Clebsch-Gordan decomposition of $V_\rho \otimes V_\lambda$ as in Eq. (6). Then*

$$T_t \Big|_{V_\rho \otimes V_\lambda} = |\lambda + \rho \pm \gamma(3t - 1)|^2 + (3t - 1)(3t - 2)(|\rho|^2 - |\gamma|^2). \quad (7)$$

Notice that for $t = 1/3$, $T_{1/3}$ acts on $V_\rho \otimes V_\lambda$ as $|\lambda + \rho|^2$, obviating the Clebsch-Gordan decomposition.

By the injectivity of $\pi \otimes 1$, one obtains as well the action of \mathcal{D}_t^2 on $V_\rho \otimes V_\lambda$.

IV. SPECTRAL ACTION FOR $SU(2)$

From Sec. III, we reduce the problem of finding the spectrum of \mathcal{D}_t^2 to finding the Clebsch-Gordan decomposition of $V_\rho \otimes V_\lambda$. This section will focus on the specific case of $SU(2)$.

In the case of $SU(2)$, let V_m denote the irreducible representation of $SU(2)$ of dimension $m + 1$, $m \in \mathbb{N}$. The Weyl vector is given by $\rho = 1$, and the tensor product decomposition that we need for our calculation is

$$V_1 \otimes V_m = V_{m+1} \oplus V_{m-1}.$$

For $m = 0$, we ignore V_{-1} and the equation reads $V_1 \otimes V_0 = V_1$. In this case, the Clifford module \mathcal{S} equals just a single copy of V_1 .

Plugging $\lambda = m$, $\rho = 1$, and $\gamma = 1$ into Eq. (7), one obtains the action of T_t is $m + 3t$ on V_{m+1} with multiplicity $(m + 2)(m + 1)$ for $m \geq 0$; and $m + 2 - 3t$ on V_{m-1} with multiplicity $m(m + 1)$ for $m \geq 1$, which can alternatively be written as $-n - 3t$ with multiplicity $(n + 2)(n + 1)$ for $n \leq -1$ by the change of indices $m + 2 = -n$. Hence, the spectrum of \mathcal{D}_t^2 is given by $(n + 3t)^2$ for $n \in \mathbb{Z}$ with multiplicity $(n + 2)(n + 1)$. And the spectrum action $\text{Tr } f\left(\frac{\mathcal{D}_t^2}{\Lambda^2}\right)$ is

$$\text{Tr } f\left(\frac{\mathcal{D}_t^2}{\Lambda^2}\right) = \sum_{n \in \mathbb{Z}} (n + 2)(n + 1) f\left(\frac{(n + 3t)^2}{\Lambda^2}\right). \quad (8)$$

Now we follow the analysis of Chamseddine and Connes.⁴ Let $g(u) = (u + 2)(u + 1)f\left(\frac{(u + 3t)^2}{\Lambda^2}\right)$. Its Fourier transform, denoted by $\widehat{g}(x)$, is

$$\begin{aligned} \widehat{g}(x) &= \int_{\mathbb{R}} (u + 2)(u + 1) f\left(\frac{(u + 3t)^2}{\Lambda^2}\right) e^{-2\pi i x u} du \\ &= \int_{\mathbb{R}} (\Lambda y - (3t - 1))(\Lambda y - (3t - 2)) f(y^2) e^{-2\pi i x (\Lambda y - 3t)} \Lambda dy \\ &= \Lambda^3 e^{-2\pi i x (-3t)} \int_{\mathbb{R}} y^2 f(y^2) e^{-2\pi i x \Lambda y} dy \\ &\quad - \Lambda^2 e^{-2\pi i x (-3t)} \int_{\mathbb{R}} 3(2t - 1) y f(y^2) e^{-2\pi i x \Lambda y} dy \\ &\quad + \Lambda \int_{\mathbb{R}} (3t - 1)(3t - 2) f(y^2) e^{-2\pi i x \Lambda y} dy. \end{aligned}$$

Now let $\widehat{f}^{(m)}$ denote the Fourier transform of $y^m f(y^2)$. By the Poisson summation formula, $\sum_{\mathbb{Z}} g(n) = \sum_{\mathbb{Z}} \widehat{g}(x)$, the spectral action $\text{Tr } f \left(\frac{\mathcal{D}_t^2}{\Lambda^2} \right)$ (8) then becomes

$$\begin{aligned} \text{Tr } f \left(\frac{\mathcal{D}_t^2}{\Lambda^2} \right) &= \sum_{\mathbb{Z}} \widehat{g}(n) \\ &= \Lambda^3 \sum_{\mathbb{Z}} e^{-2\pi i n(-3t)} \widehat{f}^{(2)}(\Lambda n) \\ &\quad - \Lambda^2 \sum_{\mathbb{Z}} 3(2t-1) e^{-2\pi i n(-3t)} \widehat{f}^{(1)}(\Lambda n) \\ &\quad + \Lambda \sum_{\mathbb{Z}} (3t-1)(3t-2) e^{-2\pi i n(-3t)} \widehat{f}(\Lambda n). \end{aligned}$$

By taking f to be a Schwartz function, $\widehat{f}^{(m)}$ has rapid decay, thus for all k

$$|\widehat{f}^{(m)}(\Lambda n)| < C_k (\Lambda n)^{-k}$$

and

$$\left| \sum_{n \neq 0} \widehat{f}^{(m)}(\Lambda n) \right| < C'_k \Lambda^{-k}.$$

As a result,

$$\sum_{n \neq 0} \widehat{g}(n) \in O(\Lambda^{-\infty}).$$

Finally, we obtain

Theorem 4.1: *The spectral action of \mathcal{D}_t for $SU(2)$ is*

$$\begin{aligned} \text{Tr } f \left(\frac{\mathcal{D}_t^2}{\Lambda^2} \right) &= \widehat{g}(0) + O(\Lambda^{-\infty}) \\ &= \Lambda^3 \widehat{f}^{(2)}(0) - \Lambda^2 \widehat{f}^{(1)}(0) 3(2t-1) + \Lambda \widehat{f}(0) (3t-1)(3t-2) + O(\Lambda^{-\infty}) \\ &= \Lambda^3 \int_{\mathbb{R}} y^2 f(y^2) dy + \Lambda (3t-1)(3t-2) \int_{\mathbb{R}} f(y^2) dy + O(\Lambda^{-\infty}). \end{aligned}$$

The result of Theorem (8) coincides with that of Ref. 4 for $t = 1/2$, where $\mathcal{D}_{1/2}$ is the spin-Dirac operator.

Observe that for the cubic Dirac operator ($t = 1/3$), the coefficient of the Λ term is zero. As $SU(2)$ is three dimensional, this coefficient is the second Seeley-De Witt coefficient, which depicts the generalized Einstein-Hilbert action. Therefore, the cubic Dirac operator gives an identically vanishing Einstein-Hilbert action.

V. SPECTRUM OF DIRAC LAPLACIAN OF $SU(3)$

First, let us summarize our results for the spectrum of the Dirac Laplacian of $SU(3)$.

Theorem 5.1: *In each row of the table below, for each pair (p, q) with p, q in the set of parameter values displayed, the Dirac Laplacian \mathcal{D}_t^2 of $SU(3)$ has an eigenvalue in the first column of the multiplicity listed in the center column.*

Let

$$\lambda(u, v) = u^2 + v^2 + uv,$$

and

$$m(a, b) = \frac{(p+1)(q+1)(p+q+2)(p+1+a)(q+1+b)(p+q+2+a+b)}{4}.$$

We denote by $\mathbb{N}^{\geq a}$, the set $\{n \in \mathbb{N} : n \geq a\}$, and we take \mathbb{N} to be the set of integers greater than or equal to zero.

| Eigenvalue | Multiplicity | Parametervalue |
|-----------------------------------|--------------|--|
| $\lambda(p+3t, q+3t)$ | $m(1, 1)$ | $p \in \mathbb{N}, q \in \mathbb{N}$ |
| $\lambda(p+2-3t, q-1+6t)$ | $m(-1, 2)$ | $p \in \mathbb{N}, q \in \mathbb{N}$ |
| $\lambda(p+1, q+1)+3(3t-1)(3t-2)$ | $m(0, 0)$ | $p \in \mathbb{N}, q \in \mathbb{N}, (p, q) \neq (0, 0)$ |
| $\lambda(p+3-6t, q+3t)$ | $m(-2, 1)$ | $p \in \mathbb{N}^{\geq 1}, q \in \mathbb{N}$ |
| $\lambda(p-1+6t, q+2-3t)$ | $m(2, -1)$ | $p \in \mathbb{N}, q \in \mathbb{N}$ |
| $\lambda(p+1, q+1)+3(3t-1)(3t-2)$ | $m(0, 0)$ | $p \in \mathbb{N}^{\geq 1}, q \in \mathbb{N}^{\geq 1}$ |
| $\lambda(p+3t, q+3-6t)$ | $m(1, -2)$ | $p \in \mathbb{N}, q \in \mathbb{N}^{\geq 1}$ |
| $\lambda(p+2-3t, q+2-3t)$ | $m(-1, -1)$ | $p \in \mathbb{N}, q \in \mathbb{N}$ |

There is some flexibility in the set of parameter values. For instance in the second line, we could have used instead $p \in \mathbb{N}^{\geq 1}$ since for that line the multiplicity is zero whenever $p = 0$.

A. Spectrum for $t = 1/3$

In the case of $t = 1/3$, the expression of the spectrum becomes much simpler, as we no longer need to take the Clebsch-Gordan decomposition into account.

Theorem 5.2: *The spectrum for the Dirac Laplacian $\mathcal{D}_{1/3}^2$ of $SU(3)$ is given in the following table.*

| Eigenvalue | Multiplicity | Parametervalue |
|------------------|------------------|--------------------------------------|
| $p^2 + q^2 + pq$ | $2p^2q^2(p+q)^2$ | $p \in \mathbb{N}, q \in \mathbb{N}$ |

We will later apply the Poisson summation formula to the result of Theorem 5.2, we will make use of the nice property that the multiplicities of $(p, 0)$ and $(0, q)$ are zero for $p, q \in \mathbb{N}$.

B. Derivation of the spectrum

The pairing $\langle \cdot, \cdot \rangle$ is in general the dual pairing on the weight space of a nondegenerate symmetric bilinear form on the Cartan subalgebra of \mathfrak{g} . Such a nondegenerate symmetric bilinear form is necessarily a constant multiple of the Killing form, which for $SU(3)$ is given by

$$\kappa(X, Y) = 6 \operatorname{Tr}(\operatorname{ad}_X \operatorname{ad}_Y), \quad (9)$$

where the trace and multiplication are taken in the natural representation of X, Y as 3×3 matrices. One may identify \mathfrak{g}^* with \mathfrak{g} by identifying $\lambda \in \mathfrak{g}^*$ with the unique X_λ such that $\langle X_\lambda, Y \rangle = \lambda(Y)$, for all $Y \in \mathfrak{g}$. This is possible due to the nondegeneracy of the pairing on \mathfrak{g} . In this way, one defines the dual pairing on \mathfrak{g}^* . The particular pairing which occurs depends on the normalization of the Riemannian metric. More specifically, the pairing is related to the Casimir operator by (5), the Casimir operator in turn depending on the normalization of the Riemannian metric.

Henceforth, we assume that the metric is normalized so that $\langle \rho, \rho \rangle = 3$. This leads to the simplest expressions for the spectrum.

In order to derive the spectrum of the Dirac Laplacian, one must first analyze the pairing of weights. We take for our basis of the Cartan subalgebra, \mathfrak{h} , the set $\{H_1, H_2\}$,

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (10)$$

We identify weights concretely using this basis. I.e. for $\lambda \in \mathfrak{h}^*$, we identify λ with $(\lambda(H_1), \lambda(H_2))$.

In terms of Theorem 3.6, in the case of $SU(3)$ we have $\lambda = (p, q) \in \mathbb{N} \times \mathbb{N}$, $\rho = (1, 1)$, $\gamma = (a, b) = (0, 0), (1, 1), (2, -1)$, or $(-1, 2)$; and its multiplicity is $\frac{1}{4}(p+1 \pm a)(p+1)(q+1 \pm b)(q+1)(p+q+2 \pm (a+b))(p+q+2)$. And $\mathcal{S} = V_{(1,1)}$.

The weights $\lambda_1 = (1, 0)$ and $\lambda_2 = (0, 1)$ form an \mathbb{N} -basis of the highest weights of irreducible representations of $SU(3)$. The pairing of weights can be determined up to normalization, using duality, and the Killing form, from which one deduces the relations

$$\langle \lambda_1, \lambda_1 \rangle = \langle \lambda_2, \lambda_2 \rangle = 2\langle \lambda_1, \lambda_2 \rangle. \quad (11)$$

From these relations and Lemma 3.4, we immediately obtain the following lemma.

Lemma 5.3: On the irreducible representation of highest weight (p, q) , $p, q \in \mathbb{N}$, the Casimir element acts by the scalar

$$\text{Cas} \Big|_{V_{(p,q)}} = (p^2 + q^2 + 3p + 3q + pq) \langle \lambda_1, \lambda_1 \rangle. \quad (12)$$

For the normalization that we are considering, we have $\langle \lambda_1, \lambda_1 \rangle = 1$.

We have listed the irreducible representations of $SU(3)$ as well as the action of the Casimir operator on them. To write down the spectrum of the Dirac Laplacian the only obstacle now is to understand the term ΔCas in Theorem 3.6; i.e. we need to know the Clebsch-Gordan coefficients of the tensor products $V_\rho \otimes V_{(p,q)}$. These were computed in Ref. 11. We recall the Clebsch-Gordan coefficients that we will need below.

Lemma 5.4 (Ref. 11): The decomposition of $V_\rho \otimes V_{(p,q)}$ into irreducible representations is

$$V_\rho \otimes V_{(p,q)} = \bigoplus_\mu V_\mu, \quad (13)$$

where the summands V_μ appearing in the direct sum are given by the following table:

| Summand | ParameterValues |
|-----------------|--|
| $V_{(p+1,q+1)}$ | $p \in \mathbb{N}, q \in \mathbb{N}$ |
| $V_{(p-1,q+2)}$ | $p \in \mathbb{N}^{\geq 1}, q \in \mathbb{N}$ |
| $V_{(p,q)}$ | $p \in \mathbb{N}, q \in \mathbb{N}, (p, q) \neq (0, 0)$ |
| $V_{(p-2,q+1)}$ | $p \in \mathbb{N}^{\geq 2}, q \in \mathbb{N}$ |
| $V_{(p+2,q-1)}$ | $p \in \mathbb{N}, q \in \mathbb{N}^{\geq 1}$ |
| $V_{(p,q)}$ | $p \in \mathbb{N}^{\geq 1}, q \in \mathbb{N}^{\geq 1}$ |
| $V_{(p+1,q-2)}$ | $p \in \mathbb{N}, q \in \mathbb{N}^{\geq 2}$ |
| $V_{(p-1,q-1)}$ | $p \in \mathbb{N}^{\geq 1}, q \in \mathbb{N}^{\geq 1}$ |

(14)

Each summand in the left column appears once if (p, q) lies in the set of parameter values listed on the right column. For instance for $(p, q) = (1, 1)$, the summand $V_{(p,q)} = V_{(1,1)}$ appears twice in the direct sum decomposition, since $V_{(p,q)}$ appears twice in the left column, and (p, q) is in the set of parameter values in each of the two rows.

By combining Theorem 3.6, Lemma 5.3, and Lemma 5.4, we obtain Theorem 5.1. The multiplicities are obtained using the Weyl dimension formula

$$\dim V_{(p,q)} = \frac{1}{2}(p+1)(q+1)(p+q+2). \quad (15)$$

When $t = 1/3$, the formula for the Dirac Laplacian in Theorem 3.6 simplifies to

$$\mathcal{D}_{1/3}^2 = 1 \otimes \text{Cas} + 3. \quad (16)$$

Therefore, we no longer need to decompose any tensor products into irreducible components, and using just Lemma 5.3, one obtains Theorem 5.2.

VI. SPECTRAL ACTION FOR $SU(3)$

In this section, we compute the spectral action, $\text{Tr } f(\mathcal{D}_t^2/\Lambda^2)$. In the case $t = 1/3$, one may apply the Poisson summation formula as in Ref. 4 to quickly obtain the full asymptotic expansion for the spectral action. For general t however, this approach no longer works. An expansion can however still be generated using a two variable generalization of the Euler-Maclaurin formula.⁶ However, this requires more work to produce, and produces the full expansion of the spectral action if the test function f is taken to be “flat” at the origin. The flatness assumption of f is natural as the role of f is to act as a cut-off function. Here, we compute the spectral action to order Λ^0 .

A. $t = 1/3$

Let $f \in \mathcal{S}(\mathbb{R})$ be a Schwarz function. By Theorem 5.2, the spectral action of $SU(3)$, for $t = 1/3$ is given by

$$\text{Tr } f(\mathcal{D}_{1/3}^2/\Lambda^2) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} 2p^2 q^2 (p+q)^2 f\left(\frac{p^2 + q^2 + pq}{\Lambda^2}\right). \quad (17)$$

In order to apply the Poisson summation formula, one needs to turn this sum into a sum over \mathbb{Z}^2 . For this purpose, one takes advantage of the fact that the expressions for the eigenvalues and multiplicities are both invariant under a set of transformations of \mathbb{N}^2 which together cover \mathbb{Z}^2 . The linear transformations of \mathbb{N}^2 which together cover \mathbb{Z}^2 are

$$\begin{aligned} T_1(p, q) &= (p, q), \\ T_2(p, q) &= (-p, p+q), \\ T_3(p, q) &= (-p-q, p), \\ T_4(p, q) &= (-p, -q), \\ T_5(p, q) &= (p, -p-q), \\ T_6(p, q) &= (p+q, -p). \end{aligned}$$

Each of the transformations is injective on \mathbb{N}^2 . The union of the images is all of \mathbb{Z}^2 . The six images of \mathbb{N}^2 overlap on the sets $\{(p, q): p = 0\}$ and $\{(p, q): q = 0\}$. However, the multiplicity is equal to zero at these points, and so this overlap is of no consequence. Therefore, we may now write the spectral action as a sum over \mathbb{Z}^2 as

$$\text{Tr } f(\mathcal{D}_{1/3}^2/\Lambda^2) = \frac{1}{6} \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} 2p^2 q^2 (p+q)^2 f\left(\frac{p^2 + q^2 + pq}{\Lambda^2}\right). \quad (18)$$

For a sufficiently regular function, the Poisson summation formula (in two variables) is

$$\sum_{\mathbb{Z}^2} g(p, q) = \sum_{\mathbb{Z}^2} \widehat{g}(x, y). \quad (19)$$

Applying Eqs. (19) to (18), and applying the argument used in Ref. 4 we get the following result.

Theorem 6.1: *Let $f \in \mathcal{S}(\mathbb{R})$ be a Schwarz function. For $t = 1/3$, the spectral action of $SU(3)$ is*

$$\text{Tr } f(\mathcal{D}_{1/3}^2/\Lambda^2) = \frac{1}{3} \iint_{\mathbb{R}^2} x^2 y^2 (x+y)^2 f(x^2 + y^2 + xy) dx dy \Lambda^8 + O(\Lambda^{-k}), \quad (20)$$

for any integer k .

B. General t and the Euler-Maclaurin formula

The one-variable Euler-Maclaurin formula was used in Ref. 5 to compute the spectral action of $SU(2)$ equipped with the Robertson-Walker metric. A two-variable Euler-Maclaurin formula may be applied here to compute the spectral action on $SU(3)$ for all values of t .

Let m be a positive integer. Let g be a function on \mathbb{R}^2 with compact support. One instance of the two-variable Euler-Maclaurin formula is⁶

$$\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} 'g(p, q) = L^{2k} \left(\frac{\partial}{\partial h_1} \right) L^{2k} \left(\frac{\partial}{\partial h_2} \right) \int_{h_1}^{\infty} \int_{h_2}^{\infty} g(p, q) dp dq \Big|_{h_1=0, h_2=0} + R_m^{st}(g). \quad (21)$$

The notation $\sum \sum'$ indicates that terms of the form $g(0, q)$, $q \neq 0$, and $g(p, 0)$, $p \neq 0$ have a coefficient of $1/2$, $g(0, 0)$ has a coefficient of $1/4$, and the rest of the terms are given the usual coefficient of 1. The operator $L^{2k}(S)$ is defined to be

$$L^{2k}(S) = 1 + \frac{1}{2!} b_2 S^2 + \dots + \frac{1}{(2k)!} b_{2k} S^{2k}, \quad (22)$$

where b_j is the j th Bernoulli number. The number k is defined by $k = \lfloor m/2 \rfloor$. The remainder $R_m^{st}(g)$ is

$$R_m^{st}(g) = \sum_{I \subsetneq \{1, 2\}} (-1)^{(m-1)(2-|I|)} \prod_{i \in I} L^{2k} \left(\frac{\partial}{\partial h_i} \right) \int_{h_1}^{\infty} \int_{h_2}^{\infty} \prod_{i \notin I} P_m(x_i) \prod_{i \notin I} \left(\frac{\partial}{\partial x_i} \right)^m g(x_1, x_2) dx_1 dx_2 \Big|_{h=0}. \quad (23)$$

Equation (21) is proved in an elementary way in Ref. 6, by casting the one-variable Euler-Maclaurin formula in a suitable form, and then iterating it two times.

Using Theorem 5.1, one may write the spectral action in terms of eight summations of the form

$$\sum_{(p, q) \in \mathbb{N}^2} g_i(p, q),$$

where

$$g_i(p, q) = f \left(\frac{\lambda_i(p, q)}{\Lambda^2} \right) m_i(p, q), \quad i = 1, \dots, 8.$$

The notations $\lambda_i(p, q)$ and $m_i(p, q)$ denote the eigenvalues and multiplicities of the spectrum in Theorem 5.1.

One then applies the two-variable Euler-Maclaurin formula to each of the eight summations to replace the sums with integrals. Then to obtain an asymptotic expression in Λ , one controls the remainder, $R_m^{st}(g)$, to arbitrary order in Λ by taking m to be sufficiently large, and computes the big- O behavior of the other integrals to arbitrary order in Λ by applying the multivariate Taylor's theorem to a large enough degree. The terms in the Taylor expansions of the integrals yield the asymptotic expansion of the spectral action.

C. Analysis of remainders

Let us consider in detail the case $I = \{\}$, of the remainder, (23). The functions $P_m(x_i)$ are periodic, and hence bounded. Furthermore, they are independent of Λ . Therefore to study the big- O behavior with respect to Λ of the remainder, (23), we only need to estimate the integral

$$\iint \left| \frac{\partial^m}{\partial p^m} \frac{\partial^m}{\partial q^m} f(s\lambda(p, q)) m(p, q) \right|. \quad (24)$$

The integration happens over $(\mathbb{R}^+)^2 = [0, \infty) \times [0, \infty)$, and $m(p, q)$ is the multiplicity polynomial. The differentiated function is a sum of terms, whose general term is given by

$$C s^i f^{(i)}(t\lambda) \lambda^{(a_1, b_1)}(p, q) \dots \lambda^{(a_i, b_i)}(p, q) m^{(j, k)}(p, q), \quad (25)$$

where C is a combinatorial constant, $s = \Lambda^{-2}$, and where j and k are less than or equal to m and $0 \leq i \leq 2m - j - k$ and

$$\sum (a_i, b_i) = (m - j, m - k).$$

Since m is degree 4 in both p and q , we know that $j \leq 4$ and $k \leq 4$. Since λ is degree 2 in both p and q we know that each of the coefficients a_k, b_k is less than or equal to 2. Therefore, one has the estimate

$$2i \geq \sum a_i = m - j \geq m - 4, \quad (26)$$

and so

$$i \geq \frac{m - 4}{2}. \quad (27)$$

It is not too hard to see that the integral

$$\iint f^{(i)}(s\lambda) \lambda^{(a_1, b_1)}(p, q) \dots \lambda^{(a_i, b_i)}(p, q) m^{(j, k)}(p, q) dp dq \quad (28)$$

is uniformly bounded as s approaches zero, and therefore we have that the integral has a big- O behavior of $O(s^{\frac{m-4}{2}})$ as s goes to zero.

The same argument gives the same estimate for the terms in the cases $I = \{1\}$ and $I = \{2\}$. Therefore we have shown

Lemma 6.2: The remainder $R_m^{st}(g)$ behaves like $O(\Lambda^{-(m-4)})$ as Λ approaches infinity.

Since the sum in the Euler-Maclaurin formula, (21) gives only a partial weight to terms on the boundary, and since the functions $g_i(p, q)$, are at times nonzero on the boundary, $\{p = 0\} \cup \{q = 0\}$ even when there are no eigenvalues there, we must compensate at the boundary in order to obtain an accurate expression for the spectral action.

In doing so, one considers sums of the form

$$\sum_{p=0}^{\infty} g_i(p, 0) \quad \text{and} \quad \sum_{q=0}^{\infty} g_i(0, q).$$

One treats these sums using the usual one-variable Euler-Maclaurin formula, which for a function, h , with compact support is

$$\sum_{p=0}^{\infty} h(p) = \int_0^{\infty} h(x) dx + \frac{1}{2} h(0) - \sum_{j=1}^m \frac{b_{2j}}{(2j)!} h^{(2j-1)}(0) + R_m(h), \quad (29)$$

where the remainder is given by

$$R_m(h) = \int_0^{\infty} P_m(x) \left(\frac{\partial}{\partial x} \right)^m h(x) dx. \quad (30)$$

The necessary estimate for the remainder (30) is as follows.

Lemma 6.3: $R_m(g(p, \cdot))$ and $R_m(g(\cdot, q))$ behave as $O(\Lambda^{-m+4})$ as Λ approaches infinity.

To prove this, we observe that since the polynomial $P_m(x)$ is bounded and independent of x , we only need to estimate for instance

$$\left| \int_0^{\infty} \left(\frac{\partial}{\partial x} \right)^m g_i(x, 0) dx \right|.$$

The function $g_i(x, 0)$ is of the form

$$f \left(\frac{ax^2 + bx + c}{\Lambda^2} + d \right) m(x, 0), \quad (31)$$

where a, b, c are independent of Λ and x , and d is independent of x . The polynomial $m(x, 0)$ is of degree 4 in x . Therefore, when one expands the derivative of (31) using the product rule, the derivatives of $f(\frac{ax^2+bx+c}{\Lambda^2} + d)$ are all of order $j \geq m - 4$. A simple inductive argument shows that the expansion of $(\partial/\partial x)^j f(\frac{ax^2+bx+c}{\Lambda^2} + d)$ under the chain rule the terms are all of the form

$$\frac{1}{\Lambda^k} f^{(i)}\left(\frac{ax^2+bx+c}{\Lambda^2} + d\right) \alpha(x),$$

where $k \geq j$, and $\alpha(x)$ is a polynomial. Finally we conclude the proof of the lemma by observing that

$$\int_0^\infty \left(\frac{\partial}{\partial x}\right)^j f\left(\frac{ax^2+bx+c}{\Lambda^2} + d\right) \alpha(x) dx \quad (32)$$

is uniformly bounded as Λ goes to infinity.

D. Analysis of main terms

With the remainders taken care of, one still needs to work out the big- O behavior of the spectral action with respect to Λ of the remaining terms coming from the two-variable and one-variable Euler-Maclaurin formulas.

The calculation required is lengthy, but the technique is elementary. One changes variables to remove (most of) the Λ dependence from the argument of the test function f . Then, one uses Taylor's theorem to remove the Λ dependence from the limits of integration, and whatever Λ dependence remains in the argument of f . In this way, one can obtain the big- O behavior of the spectral action with respect to Λ to any desired order. We have performed the computation up to constant order in Λ . If one assumes that the test function f has all derivatives equal to zero at the origin, then one obtains the asymptotic expansion to all orders in Λ .

To give a better idea of how the calculation proceeds, let us consider in detail a couple of terms coming from the Euler-Maclaurin formulas.

One term that appears upon application of the Euler-Maclaurin formula is

$$\int_0^\infty \int_0^\infty g_1(p, q) dp dq, \quad (33)$$

where

$$g_1(p, q) = f\left(\frac{(p+3t)^2 + (q+3t)^2 + (p+3t)(q+3t)}{\Lambda^2}\right) \\ \times \frac{(p+1)(q+1)(p+q+2)(p+2)(q+2)(p+q+4)}{4}$$

First, one performs on (33) the change of variables,

$$x = \frac{p+3t}{\Lambda} \quad \text{and} \quad y = \frac{q+3t}{\Lambda},$$

whereby one obtains

$$\frac{1}{4} \int_{3t/\Lambda}^\infty \int_{3t/\Lambda}^\infty f(x^2 + y^2 + xy)(1-3t+x\Lambda)(2-3t+x\Lambda) \times \\ \times (1-3t+y\Lambda)(2-3t+y\Lambda)(2-6t+x\Lambda+y\Lambda)(4-6t+x\Lambda+y\Lambda) \Lambda^2 dx dy.$$

Next, one does a Taylor expansion on the two lower limits of integration about 0. The first term in this Taylor series is obtained by setting the limits of integration to zero.

$$\frac{1}{4} \int_0^\infty \int_0^\infty f(x^2 + y^2 + xy)(1-3t+x\Lambda)(2-3t+x\Lambda) \times \\ (1-3t+y\Lambda)(2-3t+y\Lambda)(2-6t+x\Lambda+y\Lambda)(4-6t+x\Lambda+y\Lambda) \Lambda^2 dx dy. \quad (34)$$

Remarkably, if one sums the analog of (34) for g_1, \dots, g_8 one obtains the complete spectral action to constant order. All of the other terms which appear in the computation (of which there are many) cancel out, to constant order in Λ , in an intricate manner.

The end result of the calculation is the following.

Theorem 6.4: *Let f be a real-valued function on the real line with compact support. To constant order, the spectral action, $\text{Tr } f(\mathcal{D}_t^2/\Lambda^2)$ of $SU(3)$ is equal to*

$$\begin{aligned} & \text{Tr } f(\mathcal{D}_t^2/\Lambda^2) \\ &= 2 \iint_{(\mathbb{R}^+)^2} f(x^2 + y^2 + xy) x^2 y^2 (x + y)^2 dx dy \Lambda^8 \\ &+ 3(3t - 1)(3t - 2) \iint_{(\mathbb{R}^+)^2} f(x^2 + y^2 + xy) (x^4 + 2x^3 y + 3x^2 y^2 + 2xy^3 + y^4) dx dy \Lambda^6 \\ &+ 9(3t - 1)^2(3t - 2)^2 \iint_{(\mathbb{R}^+)^2} f(x^2 + y^2 + xy) (x^2 + xy + y^2) dx dy \Lambda^4 \\ &+ 6(3t - 1)^3(3t - 2)^3 \iint_{(\mathbb{R}^+)^2} f(x^2 + y^2 + xy) dx dy \Lambda^2 + O(\Lambda^{-1}). \end{aligned}$$

Here, the integrals are taken over the set $(\mathbb{R}^+)^2 = [0, \infty) \times [0, \infty)$. When f is taken to be a cut-off function so that it is flat at the origin, this expression gives the full asymptotic expansion of the spectral action.

The linear transformations, $T_1 \dots T_6$ in Subsection VI A, are all unimodular, and the images of $(\mathbb{R}^+)^2$ cover \mathbb{R}^2 , up to a set of measure zero. Therefore, in the case of $t = 1/3$, integrating over \mathbb{R}^2 multiplies the result by a factor of 6, and we see that Theorem 6.4 agrees with Theorem 6.1.

When computing the asymptotic expansion of the spectral action using the Euler-Maclaurin formula, as a result of the chain rule, the negative powers, Λ^{-j} appear only with derivatives $f^{(k)}(0)$, $k \geq j$. This is why the terms of the asymptotic expansion vanish for negative powers of Λ , when the derivatives of f vanish at zero.

VII. SLOW ROLL POTENTIAL AND COSMIC TOPOLOGY

The above non-perturbative computation of the spectral action for the one-parameter family of Dirac operators gives an expression which naturally lends itself to the consideration of the inflation model studied in the recent papers.^{3,12,13}

In order to generate an inflation scenario from the non-perturbative spectral action, one computes the non-perturbative spectral action of the Friedmann metric on the product of S^3 with radius a and a compactified time dimension in the form of a circle with radius β . The one-parameter family of Dirac operators above makes available for study a one-parameter family of Dirac operators on the product space-time. This inflation is a toy model insofar that the geometry considered does not include the matter sector. It is the same model as that of Ref. 12. A more sophisticated analysis would require consideration of an almost commutative geometry, including an analysis of the resulting twisted Dirac operators, for instance as in Ref. 3.

In the model that we consider, the family of Dirac operators on the product space-time is given by

$$\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}_t \otimes 1 + i \otimes \mathcal{D}_s \\ \mathcal{D}_t \otimes 1 - i \otimes \mathcal{D}_s & 0 \end{pmatrix},$$

where \mathcal{D}_s is the Dirac operator on S^1 acting on the set of L^2 -sections $\{f : \mathbb{R} \rightarrow \mathbb{C} | f(x+1) = e^{2\pi i s} f(x)\}$ of the complex line bundle L_s . The spectrum of \mathcal{D}_s is $\{\mathbb{Z} + s\}$ with each point having multiplicity 1. The square is then given by

$$\mathcal{D}_{(t,s)}^2 = \begin{pmatrix} \mathcal{D}_t^2 \otimes 1 + 1 \otimes \mathcal{D}_s^2 & 0 \\ 0 & \mathcal{D}_t^2 \otimes 1 + 1 \otimes \mathcal{D}_s^2 \end{pmatrix}.$$

One may compute the spectral action in its non-perturbative form for the above Dirac operator using the Poisson summation formula in two dimensions. First observe that the eigenvalues of $\mathcal{D}_{(t,s)}^2/\Lambda^2$ are

$$(n + 3t)^2/(\Lambda a)^2 + (m + s)^2/(\Lambda \beta)^2,$$

and for $n, m \in \mathbb{Z}$, the eigenvalue appears with multiplicity $2(n + 1)(n + 2)$. We define

$$g(u, v) = 2(u - (3t - 1))(u - (3t - 2))h(u^2/(\Lambda a)^2 + v^2/(\Lambda \beta)^2),$$

so that the spectral action becomes

$$\text{Tr } h(\mathcal{D}_{(t,s)}^2/\Lambda^2) = \sum_{(n,m) \in \mathbb{Z}^2} g(n + 3t, m + s).$$

Applying the Poisson summation formula in two variables and some simple estimates, the spectral action, up to a term of order $O(\Lambda^{-k})$ for any k , is equal to $\widehat{g}(0, 0)$. Computing $\widehat{g}(0, 0)$ reveals that

$$\begin{aligned} \widehat{g}(0, 0) &= 2\Lambda\beta(\Lambda a)^3 \int_{\mathbb{R}^2} x^2 h(x^2 + y^2) dx dy \\ &\quad - 2(6t - 3)\Lambda\beta(\Lambda a)^2 \int_{\mathbb{R}^2} x h(x^2 + y^2) dx dy \\ &\quad + 2(3t - 1)(3t - 2)\Lambda\beta(\Lambda a) \int_{\mathbb{R}^2} h(x^2 + y^2) dx dy. \end{aligned}$$

By converting to a variation of polar coordinates, $r = x^2 + y^2$, one obtains the following:

Theorem 7.1: *The spectral action of $\mathcal{D}_{(t,s)}$ on the above product $S^3 \times S^1$ is equal to*

$$\text{Tr } h(\mathcal{D}_{(t,s)}^2/\Lambda^2) = \pi \Lambda^4 \beta a^3 \int_0^\infty r h(r) dr + 2\pi(3t - 1)(3t - 2)\Lambda^2 \beta a \int_0^\infty h(r) dr + O(\Lambda^{-k}),$$

for any $k \in \mathbb{Z}$.

Notice that the spectral action, $\text{Tr } h(\mathcal{D}_{(t,s)}^2/\Lambda^2)$, is independent of s , the parameter of the complex line bundle.

Next, one analyzes the change of the spectral action under the transformation $\mathcal{D}_{(t,s)}^2 \mapsto \mathcal{D}_{(t,s)}^2 + \phi^2$. Under this transformation, the spectral action gains additional terms, amounting to a slow roll potential. The analysis of each term follows that of Ref. 4 and we obtain the analogous result:

Theorem 7.2: *After applying the transformation $\mathcal{D}_{(t,s)}^2 \mapsto \mathcal{D}_{(t,s)}^2 + \phi^2$, and assuming that h is a smooth constant function on the interval $[0, c]$ and that $\phi^2/\Lambda^2 \leq c$, the spectral action on the above product $S^3 \times S^1$ is equal to*

$$\begin{aligned} \text{Tr } h((\mathcal{D}_{(t,s)}^2 + \phi^2)/\Lambda^2) &= \pi \Lambda^4 \beta a^3 \int_0^\infty r h(r) dr + 2\pi(3t - 1)(3t - 2)\Lambda^2 \beta a \int_0^\infty h(r) dr \\ &\quad - \pi \Lambda^2 \beta a^3 \int_0^\infty h(r) dr + \frac{\pi}{2} \beta a h(0) \phi^2 + \frac{\pi}{2} \beta a^3 h(0) \phi^4 \\ &\quad - 2\pi \beta a(3t - 1)(3t - 2)h(0)\phi^2 + O(\Lambda^{-k}). \end{aligned}$$

One is also interested in the more general situation, where we do not insist that h be constant on an interval or make any restrictions on the values of ϕ^2/Λ^2 . In this direction we have the following result:

Theorem 7.3: *After applying the transformation $\mathcal{D}_{(t,s)}^2 \mapsto \mathcal{D}_{(t,s)}^2 + \phi^2$, and assuming that h is of the form*

$$h(x) = P(\pi x)e^{-\pi x},$$

where P is a polynomial. We have

$$\begin{aligned} \text{Tr } h \left((\mathcal{D}_{(t,s)}^2 + \phi^2)/\Lambda^2 \right) &= \pi \Lambda^4 \beta a^3 \int_0^\infty r h(r) dr + 2\pi(3t-1)(3t-2) \Lambda^2 \beta a \int_0^\infty h(r) dr \\ &\quad + \pi \Lambda^4 \beta a^3 \mathcal{V}(\phi^2/\Lambda^2) - 2\pi \Lambda^2 \beta a(3t-1)(3t-2) \mathcal{W}(\phi^2/\Lambda^2) \\ &\quad + O(\Lambda^{-k}). \end{aligned}$$

for any k , where the functions \mathcal{V} and \mathcal{W} are given by

$$\mathcal{V}(x) = \int_0^\infty r(h(r+x) - h(r)) dr \quad \text{and} \quad \mathcal{W}(x) = \int_0^x h(r) dr.$$

The corresponding slow-roll potential is then

$$V(x) = \pi \Lambda^4 \beta a^3 \mathcal{V}(x) - 2\pi \Lambda^2 \beta a(3t-1)(3t-2) \mathcal{W}(x),$$

where $x = \phi^2/\Lambda^2$.

The first and second slow-roll parameters, respectively, ϵ and η , can be derived from the slow-roll potential according to the following relations:

$$A = \frac{1}{2} \left(\frac{V'(x)}{V(x)} \right)^2 \quad \text{and} \quad B = \left(\frac{V''(x)}{V(x)} \right)^2;$$

$$\epsilon = \frac{m_{Pl}^2}{8\pi} A \quad \text{and} \quad \eta = \frac{m_{Pl}^2}{8\pi} (B - A),$$

where m_{Pl} is the Planck mass.

The slow-roll parameters are then given by

$$\begin{aligned} \epsilon &= \frac{m_{Pl}^2}{16\pi} \left(\frac{-(\Lambda a)^2 \int_x^\infty h(r) dr - 2(3t-1)(3t-2)h(x)}{(\Lambda a)^2 \int_0^\infty r(h(r+x) - h(r)) dr - 2(3t-1)(3t-2) \int_0^x h(r) dr} \right)^2, \\ \eta &= \frac{m_{Pl}^2}{8\pi} \left(\frac{(\Lambda a)^2 h(x) - 2(3t-1)(3t-2)h'(x)}{(\Lambda a)^2 \int_0^\infty u(h(u+x) - h(u)) du - 2(3t-1)(3t-2) \int_0^x h(u) du} \right)^2 \\ &\quad - \frac{m_{Pl}^2}{16\pi} \left(\frac{-(\Lambda a)^2 \int_x^\infty h(r) dr - 2(3t-1)(3t-2)h(x)}{(\Lambda a)^2 \int_0^\infty r(h(r+x) - h(r)) dr - 2(3t-1)(3t-2) \int_0^x h(r) dr} \right)^2. \end{aligned}$$

This theory of inflation is interesting insofar as the inflation potential of the theory as well as the slow-roll parameters depend on the underlying topology, as shown in Ref. 13. Here, as in Ref. 13, the slow-roll parameters are independent of the unphysical radius, β of the compactified time dimension.

APPENDIX: DETAILS OF THE CALCULATIONS

Since the Dirac Laplacian spectrum of $SU(3)$, Theorem 5.1 is divided into eight pieces, the spectral action also naturally divides into eight pieces of the form

$$\sum_{p,q} g(p, q),$$

where

$$g(p, q) = f(t\lambda_i(p, q))m_i(p, q), \quad i = 1, \dots, 8,$$

and the index values taken on by p and q are determined by the expression of the spectrum, Theorem 5.1.

By the Euler-Maclaurin formula, we have

$$\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} 'g(p, q) = L^{2k} \left(\frac{\partial}{\partial h_1} \right) L^{2k} \left(\frac{\partial}{\partial h_2} \right) \int_{h_1}^{\infty} \int_{h_2}^{\infty} g(p, q) dp dq \Big|_{h_1=0, h_2=0} + R_k(g).$$

The apostrophe in the double sum indicates that the terms $g(p, 0)$, $p \neq 0$, and $g(0, q)$, $q \neq 0$ are taken with weight 1/2, and the term $g(0, 0)$ is taken with weight 1/4. We compensate for these weights, and arrive at the equation

$$\begin{aligned} \sum_{(p,q)} g(p, q) &= L^{2k} \left(\frac{\partial}{\partial h_1} \right) L^{2k} \left(\frac{\partial}{\partial h_2} \right) \int_{h_1}^{\infty} \int_{h_2}^{\infty} g(p, q) dp dq \Big|_{h_1=0, h_2=0} + R_k(g) \\ &+ \alpha \sum_{q=0}^{\infty} g(0, q) + \beta \sum_{p=0}^{\infty} g(p, 0) + \gamma g(0, 0), \end{aligned}$$

where the indices taken on by p and q are the appropriate ones as determined by the spectrum in Theorem 5.1, and the constants α , β lie in the set $\{-1/2, 0, 1/2\}$, as determined by the spectrum. The term $\gamma g(0, 0)$ is there to ensure one has the correct term at the corner $(p, q) = (0, 0)$. One then applies the one-dimensional Euler-Maclaurin formula to these boundary sums, to get our final formula

$$\sum_{(p,q)} g(p, q) = L^{2k} \left(\frac{\partial}{\partial h_1} \right) L^{2k} \left(\frac{\partial}{\partial h_2} \right) \int_{h_1}^{\infty} \int_{h_2}^{\infty} g(p, q) dp dq \Big|_{h_1=0, h_2=0} \quad (\text{A1})$$

$$+ \alpha L^{2k} \left(\frac{\partial}{\partial h} \right) \int_h^{\infty} g(0, q) dq + \beta L^{2k} \left(\frac{\partial}{\partial h} \right) \int_h^{\infty} g(p, 0) dp \quad (\text{A2})$$

$$+ \gamma g(0, 0) + R_k(g). \quad (\text{A3})$$

We have collected the remainders coming from the two-dimensional Euler-Maclaurin formula and the two instances of the one-dimensional Euler-Maclaurin formula into a single remainder, $R_k(g)$. We already demonstrated that $R_k(g)$ can be made to behave like $O(\Lambda^{-s})$ for any s , so long as k is chosen to be sufficiently large.

Now let us analyze this final formula term by term, and demonstrate how to transform these terms into asymptotic expressions in Λ .

1. The identity term

First we consider the term

$$\int_{h_1}^{\infty} \int_{h_2}^{\infty} g(p, q) dp dq \Big|_{h_1=0, h_2=0}, \quad (\text{A4})$$

which is in a class of its own. Let us work out one concrete example. We take

$$\begin{aligned} &g(p, q) \\ &= f \left(\frac{(p+1)^2 + (q+1)^2 + (p+1)(q+1) + 3(3t-1)(3t-2)}{\Lambda^2} \right) \frac{1}{4} (p+1)^2 (q+1)^2 (p+q+2)^2. \end{aligned}$$

We perform the change of variables $(p+1)/\Lambda = u$, $(q+1)/\Lambda = v$. Then the integral (A4) becomes

$$\Lambda^2 \int_{\frac{1}{\Lambda}}^{\infty} \int_{\frac{1}{\Lambda}}^{\infty} f \left(u^2 + v^2 + uv + \frac{3(3t-1)(3t-2)}{\Lambda^2} \right) P(u, v, \Lambda) du dv. \quad (\text{A5})$$

Here and below, P denotes a generic polynomial. To move the remaining Λ dependence outside of f , we make the Taylor expansion replacement

$$f(x+y) = f(x) + f'(x)y + f''(x)y^2/2! + f'''(x)y^3/3! + f^{(4)}(x)y^4/4! + O(y^5),$$

where

$$x = u^2 + v^2 + uv \quad \text{and} \quad y = \frac{3(3t-1)(3t-2)}{\Lambda^2}.$$

Here, y^5 is $O(\Lambda^{-10})$, which is enough to suppress the positive powers of Λ in the remaining part of the expression, if one is working up to constant order.

Next, we perform a Taylor expansion in the two lower limits of integration. This process leads to three classes of terms.

If we let $h(x, y)$ denote the double integral, where x and y are the two lower limits of integration, then the expansion is of the form

$$h(x, y) = \sum_{i+j \leq s} h^{(i,j)}(0, 0) \frac{x^i y^j}{i! j!} + O(\Lambda^{-s}).$$

This expansion naturally leads to three classes of terms

Class 1: $i = j = 0$.

This is simply the double integral (A5) with lower limits set to zero

$$\Lambda^2 \int_0^\infty \int_0^\infty (\cdots) P(u, v, \Lambda) du dv.$$

If one collects this term for each of the eight pieces of the spectrum. One obtains all of the terms which contribute to the expansion of the spectral action.

Class 2: Exactly one of i, j equals zero.

Suppose for instance that $i = 0$, then the terms in this class look like

$$\Lambda^2 \int_0^\infty (-1) \left(\frac{\partial}{\partial u} \right)^{j-1} \left((\cdots) P(u, v, \Lambda) \right) \Big|_{u=0} dv \left(\frac{1}{\Lambda} \right)^j \frac{1}{j!}. \quad (\text{A6})$$

This class of terms has non-vanishing terms up to order Λ^5 .

Class 3: Neither i nor j equals zero.

This class of terms is very straightforward to compute. They are of the form

$$\Lambda^2 \left(\frac{\partial}{\partial u} \right)^{j-1} \left(\frac{\partial}{\partial v} \right)^{i-1} \left((\cdots) P(u, v, \Lambda) \right) \Big|_{u=0, v=0} \left(\frac{1}{\Lambda} \right)^i \left(\frac{1}{\Lambda} \right)^j \frac{1}{i! j!}. \quad (\text{A7})$$

This class of terms only has non-vanishing terms to constant order or lower in Λ . The constant order term is a degree 8 polynomial in t .

When working to constant order, only small values of i and j are needed. For large values of i and j the powers of $\frac{1}{\Lambda}$ suppress the powers of Λ appearing in the remainder of the expression.

2. The terms $\frac{b_{2i}}{(2i)!} \left(\frac{\partial}{\partial h} \right)^{2i} \iint g(p, q) dp dq$

Here b_{2i} are the even Bernoulli numbers. The ones we need to compute up to constant order in Λ are $b_0 = 1$, $b_2 = 1/6$, $b_4 = -1/30$, $b_6 = 1/42$, $b_8 = -1/30$. We now consider the next set of terms in (A1). Performing the partial derivative in h gives

$$\left(\frac{\partial}{\partial h} \right)^{2i} \int_0^\infty \int_h^\infty g(p, q) dp dq \Big|_{h=0} = \int_0^\infty (-1) \frac{\partial^{2i-1}}{\partial p} g(p, q) \Big|_{p=0} dq. \quad (\text{A8})$$

Once again, let us work out one case concretely. We let

$$g(p, q) = f\left(\frac{(p+3t)^2 + (q+3t)^2 + (p+3t)(q+3t)}{\Lambda^2}\right) \text{mult}(p, q). \quad (\text{A9})$$

In this case, the expression (A8) becomes

$$\int_0^\infty (-1)^j P\left(f^{(j)}\left(\frac{(3t)^2 + (q+3t)^2 + (3t)(q+3t)}{\Lambda^2}\right), q, t, \Lambda^a\right) dq. \quad (\text{A10})$$

Here and below, the argument $f^{(j)}(x)$ in the polynomial indicates that f and some of its derivatives evaluated at x are variables of the polynomial. Next, one does the variable substitution $(q+3t)/\Lambda = v$, to get

$$\Lambda \int_{3t/\Lambda}^\infty (-1)^j P\left(f^{(j)}\left(v^2 + \frac{(3t)^2 + 3t\Lambda v}{\Lambda^2}\right), v, t, \Lambda^a\right) dv. \quad (\text{A11})$$

Finally, one makes the replacement

$$f^{(j)}(x+y) = f^{(j)}(x) + f^{(j+1)}(x)y + \dots + f^{(j+8)}(x)\frac{y^8}{8!} + O(y^9), \quad (\text{A12})$$

where $x = v^2$ and $y = ((3t)^2 + 3t\Lambda v)/\Lambda^2$. Since y^9 is $O(\Lambda^{-9})$ this is enough to suppress the other powers of Λ when working to constant order.

The final expression will be of the form

$$\Lambda \int_{3t/\Lambda}^\infty (-1)^j P(f^{(j)}(v^2), v, t, \Lambda^a) dv. \quad (\text{A13})$$

Next we do the Taylor expansion in the lower limit of the integral:

$$h(x) = \sum_{j=0} h^{(j)}(0) \frac{x^j}{j!}.$$

This leads to two classes of terms

Class 1: $j = 0$ Here one simply sets the lower limit of integration to zero.

$$\frac{b_{2i}}{(2i)!} \Lambda \int_0^\infty (-1)^j P(f^{(j)}(v^2), v, t, \Lambda^a) dv.$$

This class of terms has non-vanishing contributions up to order Λ^5 .

Class 2: $j \neq 0$ These terms are of the form

$$\frac{b_{2i}}{(2i)!} \Lambda (-1)^j \frac{\partial^{j-1}}{\partial v} P(f^{(j)}(v^2), v, t, \Lambda^a) \Big|_{v=0} \left(\frac{3t}{\Lambda}\right)^j \frac{1}{j!}.$$

This class of terms only has non-vanishing contributions no higher than constant order term in Λ . This constant order term is a polynomial in t of degree 5.

3. The terms $\frac{b_{2i}}{(2i)!} \frac{b_{2j}}{(2j)!} \left(\frac{\partial}{\partial h_1}\right)^{2i} \left(\frac{\partial}{\partial h_2}\right)^{2j} \iint g(p, q) dp dq$

These terms generate just a single class of terms, which are easy to handle. They are of the form

$$\frac{b_{2i}}{(2i)!} \frac{b_{2j}}{(2j)!} \frac{\partial^{j-1}}{\partial p} \frac{\partial^{i-1}}{\partial q} g(p, q) \Big|_{p=0, q=0}.$$

This works out to an expression of the form

$$\frac{b_{2i}}{(2i)!} \frac{b_{2j}}{(2j)!} P\left(f^{(b)}\left(\frac{\dots}{\Lambda^2}\right)\right). \quad (\text{A14})$$

In order to get an asymptotic expansion to constant order, one simply replaces the arguments of all of the $f^{(b)}$ with zero.

The resulting expression has non-vanishing contributions no higher than constant order, and this term is constant with respect to t .

4. Boundary term $\int g(0, q) dq, \int g(p, 0) dp$

In the two-dimensional Euler-Maclaurin formula, the terms corresponding to the boundary, $p = 0$ and $q = 0$ are not given full weight. In addition, there may or may not be eigenvalues with positive multiplicity at the boundary, depending on which of the eight pieces of the spectrum one is considering. Therefore, one must fill in or take away the sum at the boundary in order to obtain the full spectral action. One can do this by applying the one-dimensional Euler-Maclaurin formula.

Now let us consider the terms that come when compensating for the boundary. The first terms are of the form

$$\int_0^\infty g(0, q) dq,$$

and

$$\int_0^\infty g(p, 0) dp,$$

by symmetry in p and q , it is sufficient to consider only one of these cases. Let us consider the case $p = 0$, and take

$$g(p, q) = f\left(\frac{(p + 3t)^2 + (q + 3t)^2 + (p + 3t)(q + 3t)}{\Lambda^2}\right) \text{mult}(p, q). \quad (\text{A15})$$

Then

$$\int_0^\infty g(0, q) dq = \int_0^\infty f\left(\frac{(3t)^2 + (q + 3t)^2 + (3t)(q + 3t)}{\Lambda^2}\right) \text{mult}(0, q) dq.$$

We perform the variable substitution $(q + 3t)/\Lambda = v$ so now we have

$$\Lambda \int_{3t/\Lambda}^\infty f\left(v^2 + \frac{(3t)^2 + 3t\Lambda v}{\Lambda^2}\right) P(\Lambda, v, t) dv.$$

Next we remove the Λ dependence out of f using a Taylor expansion to get

$$\Lambda \int_{3t/\Lambda}^\infty P(f^{(b)}(v^2), \Lambda^a, v, t) dv.$$

Performing the Taylor expansion in the lower limit of integration we are led to two classes of terms

$$h(x) = \sum_{j=0} h^{(j)}(0) \frac{x^j}{j!}.$$

Class 1: $j = 0$.

Simply set the lower limit to zero, and use a Taylor expansion. This class of terms has non-vanishing contributions up to order Λ^5 .

Class 2: $j \neq 0$.

These terms have non-vanishing contributions no higher than constant order in Λ . The constant order term in Λ is a polynomial in t of degree 5.

5. The boundary terms $\frac{b_{2i}}{(2i)!} \left(\frac{\partial}{\partial h}\right)^{2i} \int g(0, q) dq$

These terms are straightforward to handle and are of the form

$$-\frac{b_{2i}}{(2i)!} \left(\frac{\partial}{\partial q}\right)^{2i-1} g(0, q) \Big|_{q=0}.$$

These terms contribute no higher than constant order in Λ . The constant order term in Λ is constant in t .

6. Corner term

Finally we have the term $\gamma g(0, 0)$. This contributes no higher than constant order in Λ , and the constant order term in Λ is constant in t .

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